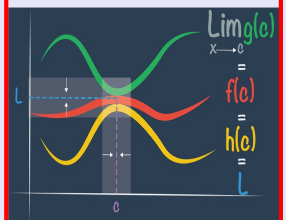
Calculus I

Lecture 17



If $\underbrace{f(x) \text{ is diff. at } a}_{f'(a) \text{ exists}}$, then $\underbrace{\text{it is Cont. at } a}_{\lim\limits_{x\to a} f(x) = f(a)}$.

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left[f(x)-f(a) + f(a)\right]$$

$$= \lim_{x\to a}\left[f(x)-f(a)\right] + \lim_{x\to a} f(a)$$

$$= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right] + \lim_{x\to a} f(a)$$

$$= \boxed{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}} \cdot \boxed{\lim_{x\to a}(x-a)} + \boxed{\lim_{x\to a} f(a)}$$

$$= f'(a)\cdot(\overset{\to 0}{a-a}) + f(a)$$

$$= f'(a)\cdot 0 + f(a) = 0 + f(a) = f(a)$$

So $\lim\limits_{x\to a} f(x) = f(a)$

So $f(x)$ is Cont. at $x = a$.

Rolle's Thrm:

1) $f(x)$ is Cont. on $[a,b]$

2) $f(x)$ is diff. on (a,b)

3) $f(a) = f(b)$

then there is at least a number C in (a,b)

Such that $f'(c) = 0$.

ex: $f(x) = \sqrt{x} - \frac{1}{3}x \quad [0,9]$

1) $f(x)$ is Cont. on $[0,9]$

2) $f(x)$ is diff. on $(0,9)$ $\quad f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

3) $f(0) = 0$, $f(9) = \sqrt{9} - \frac{1}{3}(9) = 0$, $f(0) = f(9)$ ✓

Solve $f'(c) = 0$

$$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0 \rightarrow \frac{1}{2\sqrt{c}} = \frac{1}{3}$$

$$2\sqrt{c} = 3 \qquad \sqrt{c} = \frac{3}{2}$$

$$(\sqrt{c})^2 = \left(\frac{3}{2}\right)^2 \rightarrow \boxed{c = \frac{9}{4}}$$

$\frac{9}{4} = 2.25$ is in $(0,9)$

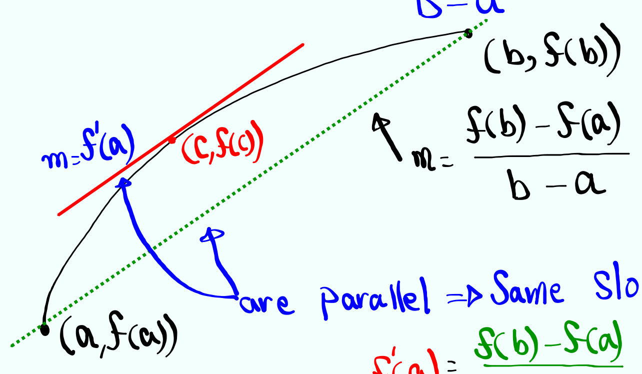
Mean - Value Theorm

1) $f(x)$ must be cont. on $[a,b]$

2) $f(x)$ must be diff. on (a,b)

Then there is at least a number C in (a,b)

Such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



are parallel $\Rightarrow$ Same slope

$$f'(a) = \frac{f(b) - f(a)}{b - a}$$

ex: $f(x) = 2x^2 - 3x + 1$, $[0,2]$

Polynomial function $\Rightarrow$ Cont. & diff. $(-\infty, \infty)$

By MVT, there is at least a number C in $(0,2)$ such that $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$

$f'(x) = 4x - 3$ $\quad f(2) = 2(2)^2 - 3(2) + 1 = 3$ $\quad f(0) = 1$

$f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$

$4c - 3 = \dfrac{3 - 1}{2 - 0}$

$4c - 3 = \dfrac{2}{2}$

$4c - 3 = 1$

$\boxed{c = 1}$

1 is in $(0,2)$ By MVT.

Check the conditions of MVT for $f(x) = x^3 - 3x + 2$ on $[-2, 2]$, then find all numbers c that satisfy the Conclusion of MVT.

1) Cont. on $[-2,2]$ ✓ 2) diff. on $(-2,2)$ ✓

Conclusion $\rightarrow$ $f'(c) = \dfrac{f(2) - f(-2)}{2 - (-2)}$

$f'(x) = 3x^2 - 3$

$f(2) = 2^3 - 3(2) + 2 = 4$

$f(-2) = (-2)^3 - 3(-2) + 2 = 0$

$3c^2 - 3 = \dfrac{4 - 0}{2 + 2}$

$3c^2 - 3 = 1$

$3c^2 = 4$

$c^2 = \dfrac{4}{3}$

$c = \pm \dfrac{2}{\sqrt{3}}$

$\approx \pm 1.155$

± 1.155 are in $(-2,2)$ By M.V.T.

$f(x) = \dfrac{x}{x-1}$

1) Domain $(-\infty, 1) \cup (1, \infty)$

$x - 1 \neq 0 \qquad x \neq 1$

2) Vertical Asymptote $\Rightarrow x = 1$

3) Horizontal Asymptote $\Rightarrow \lim\limits_{x \to \infty} f(x)$ & $\lim\limits_{x \to -\infty} f(x)$

$$\lim_{x \to \infty} \frac{x}{x-1} = \lim_{x \to \infty} \frac{\frac{x}{x}}{\frac{x}{x} - \frac{1}{x}} = \lim_{x \to \infty} \frac{1}{1 - \cancel{\frac{1}{x}}^{\,0}} = 1 \qquad \boxed{y = 1}$$

$$\lim_{x \to -\infty} \frac{x}{x-1} = 1$$

4) Y-Int $\to x = 0 \to y = \dfrac{0}{0-1} = 0 \to (0,0)$

5) X-Int $\to y = 0 \to f(x) = 0 \to \dfrac{x}{x-1} = 0 \ \boxed{x = 0}$

$(0,0)$

6) $f'(x) = \dfrac{1(x-1) - x \cdot 1}{(x-1)^2} = \dfrac{-1}{(x-1)^2}$ $\qquad f'(x) \neq 0$, $f'(x)$ und. at $x = 1$

7) $f'(x) = -(x-1)^{-2}$

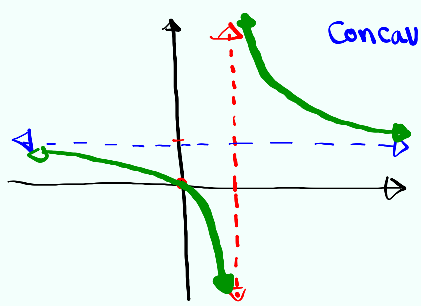
$f''(x) = -1 \cdot (-2)(x-1)^{-3} \cdot 1 = \dfrac{2}{(x-1)^3}$ $\qquad f''(x) \neq 0$, $f''(x)$ und. at $x = 1$

$f'(x) = \dfrac{-1}{(x-1)^2} \qquad f''(x) = \dfrac{2}{(x-1)^3}$

x	$-\infty$	1	∞
$f'(x)$	$-$	O	$-$
$f''(x)$	$-$	O	$+$
$f(x)$			

1 is not inflection pt.

Concavity changed by 1 is not in the domain



$f(x) = \frac{x}{x^2+1}$

1) Domain $(-\infty, \infty)$ Since $x^2+1 \neq 0$

2) V.A. $\Rightarrow$ None

3) H.A. $\Rightarrow \lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}\to 0}{1+\frac{1}{x^2}\to 0} = \frac{0}{1} = \boxed{0}$

$\lim_{x\to-\infty} \frac{x}{x^2+1} = 0$ $y=0$

4) y-Int $(0,0)$, x-Int $(0,0)$

5) even, odd, or neither?

$f(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -\frac{x}{x^2+1} = -f(x)$

$f(-x) = -f(x) \rightarrow$ odd $\rightarrow$ sym. w/t origin

6) $f'(x) = \frac{1(x^2+1) - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$f'(x)=0$
$1-x^2=0$
$x=\pm 1$

$f'(x)$ is never und.

$f'(x) = \frac{1-x^2}{(x^2+1)^2}$

7) $f''(x) = \frac{-2x(x^2+1)^2 - (1-x^2)\cdot 2(x^2+1)\cdot 2x}{[(x^2+1)^2]^2}$

$= \frac{-2x(x^2+1)^2 - 4x(1-x^2)(x^2+1)}{(x^2+1)^4}$

$= \frac{-2x\cancel{(x^2+1)}[x^2+1+2(1-x^2)]}{(x^2+1)^{\cancel{4}\,3}}$

$= \frac{-2x(3-x^2)}{(x^2+1)^3}$

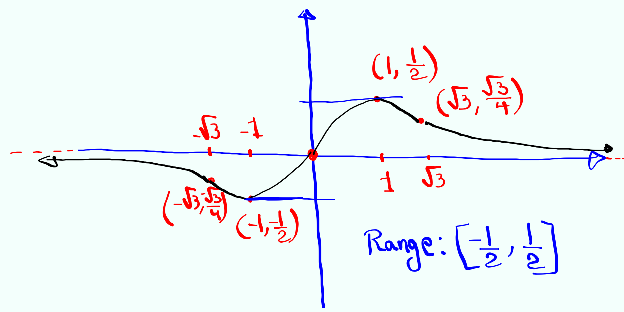
$f''(x)=0$
$-2x(3-x^2)=0$
$x=0$ $x=\pm\sqrt{3}$

$f''(x)$ is never undefined

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$$

$$f''(x) = \frac{-2x(3-x^2)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3} = \frac{2x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}$$

x	$-\infty$	$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		∞
$f'(x)$	$-$		$-$	•	$+$		$+$	•	$-$		$-$	
$f''(x)$	$-$	•	$+$		$+$	•	$-$		$-$	•	$+$	
$f(x)$	↘		↘		↗		↗		↘		↘	



Range: $[-\frac{1}{2}, \frac{1}{2}]$

Looking ahead:

$$f''(x) = -2 + \overset{6\cdot 2x}{\boxed{12x}} - \overset{4\cdot 3x^2}{\boxed{12x^2}}$$

$f(0) = 4$
$f'(0) = 12$

Find $f(x)$

$f'(x) = -2x + 6x^2 - 4x^3 + C$

$f'(0) = -2(0) + 6(0)^2 - 4(0)^3 + C = 12$

$\boxed{C = 12}$

$$f'(x) = -2x + \overset{2\cdot 3x^2}{\boxed{6x^2}} - 4x^3 + 12$$

$f(x) = -x^2 + 2x^3 - x^4 + 12x + C$

$f(0) = -0^2 + 2(0)^3 - 0^4 + 12(0) + C = 4$

$\boxed{C = 4}$

$$\boxed{f(x) = -x^2 + 2x^3 - x^4 + 12x + 4}$$

Evaluate $\lim_{x \to \infty} \frac{3x-2}{2x+1}$ $\frac{\infty}{\infty}$ I.F.

Divide by highest power of x

from deno. $x^1 = x$

$$= \lim_{x \to \infty} \frac{\frac{3x}{x} - \frac{2}{x}}{\frac{2x}{x} + \frac{1}{x}} = \lim_{x \to \infty} \frac{3 - \cancel{\frac{2}{x}}^{\to 0}}{2 + \cancel{\frac{1}{x}}^{\to 0}} = \boxed{\frac{3}{2}}$$

Evaluate $\lim_{x \to -\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5} = \frac{-\infty}{-\infty}$ I.F.

Divide by x^3

$$= \lim_{x \to -\infty} \frac{4 + \cancel{\frac{6}{x}}^{\to 0} - \cancel{\frac{2}{x^3}}^{\to 0}}{2 - \cancel{\frac{4}{x^2}}^{\to 0} + \cancel{\frac{5}{x^3}}^{\to 0}} = \frac{4}{2} = \boxed{2}$$

Evaluate $\lim_{x \to -\infty} \frac{x - 4}{\sqrt{25x^2+1}} = \frac{-\infty}{\infty}$ I.F.

Divide by x

but $\boxed{\begin{array}{ll} x \to \infty, & x = \sqrt{x^2} \\ x \to -\infty, & x = -\sqrt{x^2} \end{array}}$

$$\lim_{x \to -\infty} \frac{x-4}{\sqrt{25x^2+1}}$$

$$= \lim_{x \to -\infty} \frac{\frac{x}{x} - \frac{4}{x}}{\frac{\sqrt{25x^2+1}}{x}} = \lim_{x \to -\infty} \frac{1 - \frac{4}{x}}{\frac{\sqrt{25x^2+1}}{-\sqrt{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{1 - \frac{4}{x}}{-\sqrt{\frac{25x^2}{x^2} + \frac{1}{x^2}}} = \lim_{x \to -\infty} \frac{1 - \cancel{\frac{4}{x}}^{\to 0}}{-\sqrt{25 + \cancel{\frac{1}{x^2}}^{\to 0}}}$$

$$= \frac{1}{-\sqrt{25}} = \boxed{-\frac{1}{5}} = \boxed{-.2}$$

Evaluate $\lim_{x \to -\infty} \frac{\sqrt{9x^2 - x}}{x + 1} = \frac{\infty}{-\infty}$ I.F.

Divide by x

Since $x \to -\infty \Rightarrow x = -\sqrt{x^2}$

$$\lim_{x \to -\infty} \frac{\frac{\sqrt{9x^2 - x}}{-\sqrt{x^2}}}{\frac{x+1}{x}} = \lim_{x \to -\infty} \frac{-\sqrt{\frac{9x^2}{x^2} - \frac{x}{x^2}}}{\frac{x}{x} + \frac{1}{x}}$$

$$= -\lim_{x \to -\infty} \frac{\sqrt{9 - \cancel{\frac{1}{x}} \to 0}}{1 + \cancel{\frac{1}{x}} \to 0} = -\frac{\sqrt{9}}{1} = \boxed{-3}$$

Find $\lim_{x \to \infty} f(x)$ if

$$\frac{4x - 1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for $x > 5$.

$$\lim_{x \to \infty} \frac{4x - 1}{x} = 4$$

$$\lim_{x \to \infty} \frac{4x^2 + 3x}{x^2} = 4$$

by S.T. $\lim_{x \to \infty} f(x) = 4$.

Evaluate $\lim_{x \to \infty} \dfrac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{h \to 0} \dfrac{\sin h}{h} = \boxed{1}$

Let $h = \frac{1}{x}$

as $x \to \infty$, $h \to 0$

Evaluate $\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \infty - \infty$ I.F.

$$\lim_{x \to \infty} \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x} = \lim_{x \to \infty} \frac{\cancel{x^2} + 1 - \cancel{x^2}}{\sqrt{x^2+1} + x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} \quad \frac{1}{\infty} = 0$$

Evaluate $\lim_{x\to\infty}\left[\sqrt{x^2+2x}-x\right] = \infty-\infty$ I.F.

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+2x}-x\right)\left(\sqrt{x^2+2x}+x\right)}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}$$

$\frac{\infty}{\infty}$ I.F.

Divide by x

$$=\lim_{x\to\infty}\frac{\frac{2x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}+\frac{x}{x}}$$

$$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\cancel{\frac{2}{x}}^{\,0}}+1}=\frac{2}{1+1}=\frac{2}{2}=\boxed{1}$$

If $x=1000000$

$$\sqrt{1000000^2+2(1000000)}-1000000$$

$\approx .999995 \approx 1$

Sum of $\underbrace{\text{two positive numbers}}_{x+y=16}$ is 16.

10 & 6

Find Smallest Value of $y=16-x$

$10^2+6^2=136$

$\underbrace{\text{Sum of their squares.}}$

8, 8

$8^2+8^2=128$

9, 7

$9^2+7^2=130$

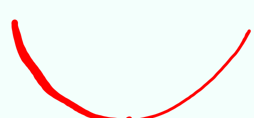
Smallest Value $\quad x^2+y^2=x^2+(16-x)^2$

$f(x)=x^2+(16-x)^2$

$f'(x)=2x+2(16-x)^1\cdot(-1)=2x-32+2x$

$=4x-32$

$f''(x)=4>0$



Min.

$f'(x)=0 \quad 4x-32=0$

$x=8$

$\boxed{8 \text{ \& } 8}$

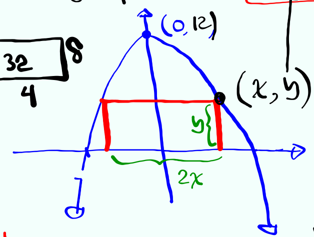
Find the dimensions of the largest area rectangle above the x-axis and one corner is on the graph of $\boxed{f(x) = 12 - x^2.}$

$x = 0 \to y = 12$

$y = 0 \to x =$

$\boxed{32} \; 8$

4



$A = LW$

$= 2xy$

$= 2x(12 - x^2)$

$f(x) = 2x(12 - x^2)$

$f(x) = 24x - 2x^3$

$f'(x) = 24 - 6x^2$

$f''(x) = -12x$

$f'(x) = 0$

$24 - 6x^2 = 0$

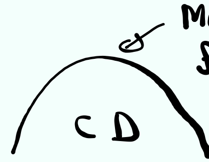
$6(4 - x^2) = 0$

$x = 2 \quad \cancel{x = -2}$

$f''(x) = 0 \quad -12x = 0 \quad \cancel{x = 0}$

x		2	
f'	+		−
f''	−		−
f			

Max



$f'(x) = 0$

$x = 2$

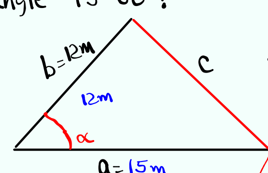
A triangle has two sides of 12m & 15m. The angle between them increases at 2°/min. How fast is the third side changing when that angle is 60°?

$\frac{d\alpha}{dt} = 2°/\text{min}$

$180° = \pi$ Rad

$1° = \frac{\pi}{180}$

$2° = \frac{\pi}{90}$



$\frac{dc}{dt} = ?$ when $\alpha = 60°$

Law of Cosines

$c^2 = a^2 + b^2 - 2ab\cos\alpha$

$c^2 = 15^2 + 12^2 - 2 \cdot 15 \cdot 12 \cos\alpha$

$c^2 = 369 - 360\cos\alpha$

$2c\frac{dc}{dt} = 0 - 360 \cdot -\sin\alpha \cdot \frac{d\alpha}{dt}$

$2c\frac{dc}{dt} = 360\sin\alpha\frac{d\alpha}{dt}$

$c^2 = 369 - 360\cos 60°$

$c^2 = 189$

$c = \sqrt{189}$

$c\frac{dc}{dt} = 180\sin\alpha\frac{d\alpha}{dt}$

$\sqrt{189}\frac{dc}{dt} = 180 \cdot \sin 60° \cdot 2°$

$\frac{dc}{dt} = \frac{\cancel{180} \cdot \frac{\sqrt{3}}{\cancel{2}} \cdot \frac{\pi}{\cancel{90}}}{\sqrt{189}}$

$= \frac{\pi\sqrt{3}}{3\sqrt{21}}$

Simplify

✓ $\boxed{.396 \text{ m/min}}$